Hodge-Witt cohomology and the slope spectral sequence

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1 Review of Hodge and de Rham cohomology

Let k be a field, and let X/k be a smooth proper variety. Recall that we have a complex of abelian sheaves, the *de Rham complex* of X/k,

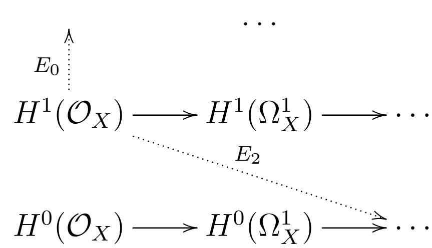
$$\mathcal{O}_X = \Omega^0_{X/k} \to \Omega^1_{X/k} \to \Omega^2_{X/k} \to \cdots \to \Omega^n_{X/k} \to 0, \tag{1}$$

where $\Omega^i_{X/k}$ is the ith exterior power of the sheaf of k-linear Kähler differentials on X and $n = \dim X$. Each term of this is a finite-rank locally free $\mathcal{O}_X$-module, but the maps are only k-linear. Taking compatible injective resolutions of the sheaves $\Omega^i_{X/k}$ gives a spectral sequence

$$E_1^{i,j} = H^j(X, \Omega^i_{X/k}) \implies \mathbb{H}^{i+j}(X, \Omega^\bullet_{X/k}) =: H^{i+j}_{\mathrm{dR}}(X/k). \tag{2}$$

Terminology: we call the k-vector spaces $H^{i,j}(X/k) := H^j(X, \Omega^i_{X/k})$ the *Hodge cohomology* of X/k; we call the right-hand side the *de Rham cohomology* of X/k; and we call this spectral sequence the *Hodge-de Rham spectral sequence.*

Its E_1-page looks like:



We note four important facts about the Hodge numbers $h^{i,j} = \dim_k H^{i,j}(X/k)$:

Fact 1 (Finiteness): Each $h^{i,j} < \infty$.

Fact 2 (Serre duality): We have $h^{i,j} = h^{n-i,n-j}$ for $n = \dim X$.

*Notes for a talk in Berkeley's student arithmetic geometry seminar. Main reference: Illusie, "Complexe de de Rham-Witt et cohomologie cristalline".

Fact 3 (Hodge symmetry): If $\operatorname{char} k = 0$, then $h^{i,j} = h^{j,i}$.

Fact 4 (Degeneration): If $\operatorname{char} k = 0$, the Hodge-de Rham spectral sequence degenerates at E_1, so in particular $\sum_{i+j=n} h^{i,j} = h^n_{\mathrm{dR}}$ for all n.

2 Review of the de Rham-Witt complex

From now on, k will be a perfect field of characteristic p, $W = W(k)$, and $\sigma : W \to W$ the Witt vector Frobenius. The de Rham-Witt complex of X/k, first constructed by Illusie in 1979, is designed to lift $\Omega^\bullet_{X/k}$ to characteristic 0, and thereby to compute crystalline cohomology. It is defined as the initial object in a rather complicated category. Rather than giving its full definition, I will just outline what kinds of structure it has, and some of the key conditions we impose. It contains the data:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \\
\downarrow R & & \downarrow R & & \\
W_2\mathcal{O}_X & \xrightarrow{d} & W_2\Omega^1_X & \xrightarrow{d} & \cdots \\
\downarrow R & & \downarrow R & & \\
W_1\mathcal{O}_X & \xrightarrow{d} & W_1\Omega^1_X & \xrightarrow{d} & \cdots
\end{array}
$$

Here each $W_n\Omega^i_X$ is a sheaf of $W_n\mathcal{O}_X$-modules, with W_nk-linear differentials and vertical quotient maps. (The bottom row is just the de Rham complex of X, and the leftmost column is the sheaf of Witt vectors of $\mathcal{O}_X$.) Additionally, each row has a multiplication map making it a cdga. Finally, each column has maps F going down and V going up, satisfying the following relations:

(a) $FV = VF = p$

(b) $dF = pFd$, $Vd = pdV$, $FdV = d$,

(c) $F(a\omega) = \sigma(a)F(\omega)$ and $V(a\omega) = \sigma^{-1}(a)V(\omega)$ for $a \in W$,

and various others.

Remark: $W_n\Omega^i_X$ can be viewed as a sheaf on the Witt scheme W_nX, which is a nilpotent thickening of X. In fact it's quasicoherent on W_nX. But we usually view it as a sheaf of W_n-modules on X, which has the same underlying topological space.

The complex $W\Omega^\bullet_X$ is defined as $\lim_{\leftarrow} W_n\Omega^\bullet_X$. The F, V, and d operators and the multiplication map pass to the inverse limit, and they have the same relations as above. Given $W\Omega^\bullet_X$ with all of these operators, we can recover $W_n\Omega^\bullet_X$ as its quotient by the images of V^n and dV^n. In practice, we pass between $W\Omega^\bullet_X$ and $(W_n\Omega^\bullet_X)_n$ more or less freely, but one must be somewhat cautious about what operations do and don't commute with the limit.

Remark: Under our smoothness hypotheses, $W\Omega^\bullet_X$ turns out to be p-torsion-free. Then each of the relations in (b) above is equivalent to saying that the map φ defined by $p^i F$ on $W\Omega^i$ commutes with d. This is useful because it means the operator φ will pass to everything in the next section, including Hodge-Witt and crystalline cohomology, and all of the maps that come up will be compatible with φ.

Theorem: The hypercohomology of the de Rham-Witt complex computes crystalline cohomology. More precisely, we have isomorphisms

$$H^*_{\text{cris}}(X/W_n) \cong \mathbb{H}^*(W_n\Omega^\bullet_X) \tag{3}$$
$$H^*_{\text{cris}}(X/W) \cong \mathbb{H}^*(W\Omega^\bullet_X). \tag{4}$$

Remark: There are some subtleties passing between these two statements. The main trick is to show that certain R^i lim's vanish by the Mittag-Leffler condition.

3 Slope spectral sequence

Definition: The *Hodge-Witt cohomology* of X/k is the sheaf cohomology $H^j(W\Omega^i_X)$.

Theorem: The W-module $H^j(W\Omega^i_X)$ is finite mod torsion, and comes with a σ-semilinear Frobenius φ. Moreover, $H^j(W_n\Omega^i_X)$ is a finite-length W_n-module, and $H^j(W\Omega^i_X) = \lim_{\leftarrow} H^j(W_n\Omega^i_X)$.

As before, we have a spectral sequence, called the *slope spectral sequence*,

$$E_1^{i,j} = H^j(W\Omega^i_X) \implies \mathbb{H}^{i+j}(W\Omega^\bullet_X) = H^{i+j}_{\text{cris}}(X/W), \tag{5}$$

computing hypercohomology. This need not degenerate at E_1, but it gives a filtration (the *slope filtration*) on each $H^n_{\text{cris}}(X/W)$ whose graded pieces are the subquotients $E_\infty^{i,j}$ of $H^j(W\Omega^i_X)$, where $i+j=n$. (The subobjects appear in the bottom right, and the quotients in top left.)

Let me briefly explain the "slope" terminology. The un-divided Frobenius φ mentioned earlier induces operators φ on each $H^j(W\Omega^i)$. These are Frobenius-semilinear maps of W-modules. Ignoring torsion, any such object has a collection of *slopes*, which are the semilinear analogues of p-adic valuations of eigenvalues.[1] Since the divided Frobenius F satisfies $FV=p$, with V topologically nilpotent, it must have all its slopes in $[0,1)$. It follows that $\varphi = p^iF$ on $H^j(W\Omega^i_X)$ has slopes in $[i, i+1)$. So if we ignore torsion and assume the spectral sequence degenerates at E_1, the induced filtration on $H^*_{\text{cris}}(X/W)$ keeps track of the slopes of φ, or more precisely the floors of the slopes. In fact this discussion implies the following theorem:

Theorem (Illusie): The slope spectral sequence degenerates at E_1 mod torsion.

[1] The literal meaning of this is given by the Dieudonné-Manin classification of Dieudonné modules over $W(\overline{k})[1/p]$.

Proof: All $E_1^{i,j}$ have φ operators with slopes in $[i, i+1)$, and all differentials respect φ. It follows that the $E_n^{i,j}$ for $n \geq 1$ inherit φ, also with slopes in $[i, i+1)$, and also commuting with differentials. But the differentials on page E_1 and beyond go between modules with no slopes in common, so they're 0 mod torsion.

Remark: We've now seen that finiteness and E_1-degeneration are true for Hodge-Witt cohomology mod torsion. Hodge symmetry and Serre duality fail even in the absence of torsion. We will see an example at the end that illustrates everything that can go wrong.

4 First examples

4.1 $\mathbb{G}_m$ and $\mathbb{A}^1$

Let's write down the de Rham-Witt complex of $X = \mathbb{G}_m = \operatorname{Spec} A$, $A = k[t^{\pm 1}]$. This is something that can be done entirely by hand, as Illusie does (and as I have done with Bhatt-Lurie-Mathew's construction). But the full calculation is a bit tedious, so I'll just give you the result along with a rough plausibility argument. I'll write down the global sections of $W\Omega_X^i$, from which one can calculate the global sections of $W_n\Omega_X^i$, and these determine the sheaf because it is quasicoherent when viewed as a sheaf on the affine scheme W_nX.

Let's first write down a reasonable guess for what $W(A)$ might look like. It should contain $W[t^{\pm 1}]$, where t is the Teichmüller lift of $t \in A$. This ring has an obvious lift of Frobenius, given by $t \mapsto t^p$ and the Witt vector Frobenius on coefficients. But the Verschiebung should send t^α to $pt^{\alpha/p}$ in order to get $FV = VF = p$. So we must adjoin $p^n t^{m/p^n}$ for all $m \in \mathbb{Z}$ and $n \geq 0$. The resulting ring is almost right, but isn't complete. In fact, $W(A)$ is equal to the V-adic completion (equivalently, coefficient-wise p-adic completion) of this ring:

$$W(A) = (W[t^{\pm 1}, p^n t^{m/p^n} : m \in \mathbb{Z}, n \geq 0])^{\widehat{}}_V \hookleftarrow W[t^{\pm 1}, p^n t^{m/p^n}] \hookleftarrow W[t^{\pm 1}] \tag{6}$$

$$F(t^\alpha) = t^{p\alpha}, \qquad \text{(and } \sigma \text{ on coefficients)} \tag{7}$$

$$V(t^\alpha) = pt^{\alpha/p}. \tag{8}$$

This is (the global sections of) $W\mathcal{O}_X$. As for $W\Omega_X^1$, we have:

$$W\Omega_X^1 = W[t^{\pm 1/p^\infty}]^{\widehat{}} \cdot \frac{dt}{t}, \tag{9}$$

$$F\left(t^\alpha \frac{dt}{t}\right) = t^{p\alpha}\frac{dt}{t}, \tag{10}$$

$$V\left(t^\alpha \frac{dt}{t}\right) = pt^{\alpha/p}\frac{dt}{t}. \tag{11}$$

(We choose dt/t as our basis to make the formulas for F and V look nicer.) The differential $d : W\mathcal{O}_X \to W\Omega_X^1$ sends t^α to $\alpha t^\alpha dt/t$, as one would expect. For future reference, we rewrite this as follows:

$$W\Omega^\bullet_{\mathbb{G}_m}(\mathbb{G}_m) = \widehat{\bigoplus}_{\alpha \in \mathbb{Z}[1/p]} (*)_\alpha \to \widehat{\bigoplus}_{\alpha \in \mathbb{Z}[1/p]} W \cdot t^\alpha \frac{dt}{t}, \tag{12}$$

where

$$(*)_\alpha = \begin{cases} W \cdot t^\alpha & \text{if } \alpha \in \mathbb{Z}, \\ p^n W \cdot t^\alpha & \text{if } \alpha = m/p^n, p \nmid m, \end{cases} \tag{13}$$

(The completion in degree 1 is a little more subtle than that in degree 0, since we are completing with respect to the image of dV^n as well as V^n. This is needed for the differential to be defined. But this won't matter for us.)

The de Rham-Witt complex of $\mathbb{A}^1$ is the same, except that in degree 0 the sum is over $\alpha \in \mathbb{Z}[1/p]^{\geq 0}$, and in degree 1 it is over $\alpha \in \mathbb{Z}[1/p]^{>0}$. (Sanity check for degree 1: $dt = t^1 dt/t$, so $V^n(dt) = p^n t^{1/p^n} dt/t$.)

4.2 Example: $\mathbb{P}^1$

Now let $X = \mathbb{P}^1$. Let's calculate the slope spectral sequence for X. To compute the E_1 page, we need to take the sheaf cohomology of each $W\Omega^i_X$. We do this using the Čech cover by $U_1 = \mathbb{P}^1 - \{0\}$ and $U_2 = \mathbb{P}^1 - \{\infty\}$. We saw above that we have:

$$W\Omega^\bullet_X(U_1 \cap U_2) = \widehat{\bigoplus}_\alpha (*)_\alpha \to \widehat{\bigoplus}_\alpha W \cdot t^\alpha \frac{dt}{t}, \tag{14}$$

$$W\Omega^\bullet_X(U_1) = \widehat{\bigoplus}_{\alpha \geq 0} (*)_\alpha \to \widehat{\bigoplus}_{\alpha > 0} W \cdot t^\alpha \frac{dt}{t}, \tag{15}$$

$$W\Omega^\bullet_X(U_2) = \widehat{\bigoplus}_{\alpha \leq 0} (*)_\alpha \to \widehat{\bigoplus}_{\alpha < 0} W \cdot t^\alpha \frac{dt}{t}. \tag{16}$$

It follows that we have:

$$H^1(W\mathcal{O}_X) = 0, \qquad H^1(W\Omega^1_X) = W \cdot \frac{dt}{t}, \tag{17}$$

$$H^0(W\mathcal{O}_X) = W, \qquad H^0(W\Omega^1_X) = 0. \tag{18}$$

Remark: One may ask why this sheaf cohomology can be computed on as Čech cohomology. One way to justify this (although probably not the only way): if we repeat the calculation with $W_n\Omega^\bullet_X$, then we're computing the sheaf cohomology of a quasicoherent sheaf on W_nX, and Čech cohomology accomplishes this. We then use the fact that $H^j(W\Omega^i_X) = \lim_\leftarrow H^j(W_n\Omega^i_X)$.

The spectral sequence clearly degenerates, so we have computed the crystalline cohomology of $\mathbb{P}^1$. We also get the Frobenius action, given by $\varphi = \sigma$ on H^0 and $\varphi = p\sigma$ on H^2.

5 Example: abelian surfaces

To illustrate the behavior of the slope spectral sequence in a more typical example, let's study what the Hodge-Witt cohomology of an abelian surface looks like.

There are three types of abelian surfaces in characteristic p, determined by the slopes of Frobenius on H^1_{cris}: either $0, 0, 1, 1$ (ordinary); $0, 1/2, 1/2, 1$; or $1/2, 1/2, 1/2, 1/2$ (supersingular).[2] Since $H^*(X) = \Lambda^* H^1(X)$ for X an abelian variety, this determines the slopes of φ on H^i: namely, the slopes on H^i are the sums of i distinct slopes on H^1. The crystalline cohomology of any abelian surface looks like $W \oplus W^4 \oplus W^6 \oplus W^4 \oplus W$, and in the supersingular case, all of the slopes on each H^i are $i/2$. This tells us exactly what the slope filtration is, which determines the description below of Hodge-Witt cohomology up to torsion.

$$\begin{matrix} k[[x]] & k[[y]] \oplus W^{\oplus 4} & W \\ W^{\oplus 4} & W^{\oplus 6} & 0 \\ W & 0 & 0 \end{matrix} \tag{19}$$

The curveball (which happens only in the supersingular case) comes in $H^2(W\mathcal{O}_X)$ and $H^2(W\Omega^1_X)$. We get infinitely much p-torsion in both of these places, killed by a nonzero differential on the E_1 page. The spectral sequence then runs out of torsion and degenerates at E_2.

[2] All of these can realized by products of two elliptic curves, depending on whether the factors are ordinary or supersingular.